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The interband light absorption coefficient in the weak disorder regime: an asymptotically exactly solvable model

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Abstract. We consider the interband light absorption coefficient (ILAC) for a d-dimensional discrete disordered system, whose Hamiltonian consist of a translation invariant part (d-dimensional discrete Laplacian) and an off-diagonal random part. Assuming that the range R of the latter is large and that its magnitude is of the order $R^{-d/2}$ we find that $R = \infty$ limit of the ILAC. We discuss some properties of the ILAC in this limit: its boundedness, edge singularities, its singular form in the limits of vanishingly translationally invariant part or infinite random part. We also show that the latter property is the same for the system with a diagonal smoothly distributed disorder, i.e. for the discrete Schrödinger operator whose random potential has a smooth probability distribution. This should be contrasted with the integrated density of states which is always smoother than the distribution of the random potential.

1. Introduction

The interband light absorption coefficient (ILAC) is an important quantitative characteristic of semiconductors. Using effective mass approximation which is widely accepted in solid state physics this quantity can be written as (Shklovskii and Éfros 1970)

$$\alpha_\Lambda(\lambda) = \frac{\alpha_0}{\omega}\frac{1}{|\Lambda|}\sum_{l,m}\delta(\lambda - E_l^+ - E_m^-)|(\psi_l^+, \psi_m^-)|^2 \tag{1}$$

provided the temperature is sufficiently low. The constant α_0 is determined by fundamental physical constants and by the band structure of the ideal (non-doped) semiconductor, ω is the light frequency, $\lambda = \hbar\omega - E_g$, where E_g is the width of the forbidden band (gap) of the ideal semiconductor and $|\Lambda|$ is the volume of the sample. Energy levels $E_j^\pm$ and wavefunctions $\psi_j^\pm$ of electrons (+) and holes (−) are the solutions of the equations

$$H^\pm\psi_j^\pm = E_j^\pm\psi_j^\pm \qquad H^\pm = -\frac{\hbar}{2m_\pm}\Delta \pm vq(x) \tag{2}$$

where $q(x)$ is a random potential generated by impurities. For simplicity we will assume that the effective masses of electrons and holes are equal and denote $\hbar/2m_\pm$ by the symbol $\mathcal{J}$.

The ILAC as all basic physical quantities characterizing macroscopic properties (the density of states (DOS), conductivity, etc) is a self-averaging quantity. This means that with probability one $\alpha_\Lambda(\lambda)$ (more precisely the respective measure) tends to a non-random limit

$\alpha(\lambda)$ as $|\Lambda| \to \infty$ (Kirsch and Pastur 1990). The ILAC is similar in several respects to the DOS. However, there are properties of the ILAC that have no analogues in the case of the DOS. For instance, the smoothness of the DOS and of the distribution of the random potential are connected rather closely. On the other hand we show below that $\alpha(\lambda)$ tends to $\delta(\lambda)$ as $\mathcal{J} \to 0$ or $v \to \infty$ for arbitrary high smoothness of the distribution of $q(x)$.

It should be noted that the smoothness or at least the boundedness of such quantities as the DOS, the conductivity, the ILAC, etc is an important element of a mathematical physics of disordered systems. However, this property has been studied extensively only for the DOS and the respective methods, the Wegner lemma (Wegner 1981) in particular, cannot be applied directly to the conductivity and the ILAC. This is because these quantities are determined via the second moments of the Green function and have more complicated analytical structure than the DOS.

Kirsch and Pastur (1990) studied the asymptotic behaviour of the ILAC for $H^{\pm}$ having the form (2) in the strong localization (strong disorder) regime $\hbar\omega \ll E_g$, i.e. for the optical transitions between levels which lie sufficiently far from the edges of the unperturbed bands. In this paper we are going to consider the weak disorder regime where the delocalized states are essential. It is widely believed that this regime can be described sufficiently well by the so-called one-site approximations (see e.g. Yonezawa and Morigaki 1973 and Lifshitz *et al* 1988). They can be considered as analogues of the mean-field approximation in statistical physics. It is well known that latter approximation is asymptotically exact in the limit of infinite interaction radius (see e.g. Hemmer and Lebowitz 1973 and Pastur and Shcherbina 1984). In the present paper we consider an analogue of this limit in the theory of disordered systems calculating the ILAC. It turns out that for random operators a natural analogue of the interaction radius is the range R of the random off-diagonal part. Thus, our class of operators is somewhat different from (2). However, our results are rather similar to those obtained for the Schrödinger operator with random potential in the framework of the coherent potential approximation (see also Khorunzhy and Pastur 1992 for discussion of this point).

The paper is organized as follows. In section 2 we describe the model and briefly discuss the connection between the Green functions and the ILAC. In section 3 we derive an expression for the ILAC in the limit $R = \infty$. In section 4 we prove the boundedness of the ILAC, the existence of an energy gap and find the behaviour of the ILAC near the gap edges. In section 5 we study the ILAC for the Schrödinger operators (2) in the limits $\mathcal{J} \to 0$ or $v \to \infty$.

2. The model

We consider multidimensional random self-adjoint operators $H_R^{\pm}$ in $l^2(\mathbb{Z}^d)$ with matrix elements:

$$H_R^{\pm}(x, y) = H_0(x - y) \pm \frac{v}{R^{d/2}} \varphi\left(\frac{|x - y|}{R}\right) W(x, y) \qquad x, y \in \mathbb{Z}^d. \tag{3}$$

Here H_0 is the lattice Laplace operator

$$(H_0\psi)(x) = d\mathcal{J}\psi(x) - \frac{\mathcal{J}}{2} \sum_{|\delta|=1} \psi(x + \delta) \tag{4}$$

which is the translational invariant part of our Hamiltonians (3), $W(x, y)$ are independent (except for the symmetry condition $W(x, y) = W(y, x)$) Gaussian random variables with zero mean and the variance of the form

$$\langle W(x,y)W(z,t)\rangle = (\delta(x-z)\delta(y-t) + \delta(x-t)\delta(y-z))$$

(here and below we denote the averaging with respect to random variables by the symbol $\langle\ldots\rangle$) and the function $\varphi(t)$, $t \in \mathbb{R}^d$, is bounded, has a compact support (say a unit ball) and

$$\int_{\mathbb{R}^d} \varphi^2(t)\,\mathrm{d}t = 1. \tag{5}$$

For fixed R the function φ restricts the disorder in (3) to a finite region in $\mathbb{Z}^d$. As $R \to \infty$, the range of disorder tends to infinity but at the same time its magnitude goes to zero. This limit is similar to the infinite interaction radius limit in statistical physics. It should be noted that other non-perturbative limits similar to those used in statistical physics can also be considered in the theory of disordered systems. We mention here the infinite number of components limit introduced by Wegner (1979) and the infinite space dimensionality limit (see e.g. Khorunzhy and Pastur 1992 and Khorunzhy *et al* 1992). In the case of the ILAC all these limits result in practically the same formulae. Therefore we consider only the infinite R limit.

To define the ILAC we consider first a finite cube $\Lambda \subset \mathbb{Z}^d$ and introduce the respective finite volume distribution function (cf (1)):

$$A_\Lambda(\lambda) = \frac{1}{|\Lambda|}\sum_{l,m} \chi(\lambda - \lambda_l^+ - \lambda_m^-)|(\psi_l^+, \psi_m^-)|^2 = \iint_{\mathbb{R}^2} \chi(\lambda - \lambda_1 - \lambda_2)\,\mathrm{d}\tau_\Lambda$$

where χ is the indicator of $(0, \infty)$ and τ_Λ is a measure on $\mathbb{R}^2$ defined as follows. For two semi-intervals Δ_1 and Δ_2, the measure of their product $\Delta = \Delta_1 \times \Delta_2$ is

$$\tau_\Lambda(\Delta) = \frac{1}{|\Lambda|}\sum_{\lambda_l^+\in\Delta_1}\sum_{\lambda_m^-\in\Delta_2} |(\psi_l^+, \psi_m^-)|^2 = \frac{1}{|\Lambda|} Tr E_\Lambda^+(\Delta_1)E_\Lambda^-(\Delta_2)$$

where $E_\Lambda^\pm$ are the resolutions of identity of the operators (3) in a finite cube Λ. According to Kirsch and Pastur (1990) if $H^\pm$ are given by (2) then there exists a non-random distribution function $A(\lambda)$ such that in all its continuity points with probability one $A_\Lambda(\lambda) \to A(\lambda)$ as $\Lambda \to \infty$. This is the self-averaging property of $A_\Lambda(\lambda)$. It can be shown that the same assertion is valid in the case of (3) for fixed R. To prove this one can use the technique developed by Pastur and Figotin (1992). The derivative of $A(\lambda)$, if it exists, we shall call the ILAC.

In terms of the resolutions of identity of $H^\pm$ the quantity $A(\lambda)$ can be written as

$$A(\lambda) = \iint_{\mathbb{R}^2} \chi(\lambda - \lambda_1 - \lambda_2)\,\mathrm{d}\tau \tag{6}$$

where

$$\tau(\Delta_1 \times \Delta_2) = \langle [E^+(\Delta_1)E^-(\Delta_2)](0,0)\rangle. \tag{7}$$

Here and below we denote the averaging over the random variables by the symbol $\langle\cdot\rangle$. If τ is absolutely continuous with respect to Lebesgue measure and $t(\lambda_1, \lambda_2)$ is its density then the ILAC can be calculated as

$$\alpha(\lambda) \equiv A'(\lambda) = \int_{\mathbb{R}} t(\lambda - \lambda_2, \lambda_2)\, d\lambda_2.$$

Let us consider the Stieltjes transform of the measure τ:

$$g(z_1, z_2) = \iint_{\mathbb{R}^2} \frac{d\tau(\lambda_1, \lambda_2)}{(\lambda_1 - z_1)(\lambda_2 - z_2)} \qquad |\operatorname{Im}|z_i| > 0.$$

By using (7) and the spectral theorem we can represent $g(z_1, z_2)$ in the form

$$g(z_1, z_2) = \langle [G^+(z_1)G^-(z_2)](0, 0)\rangle \tag{8}$$

where $G^{\pm}(z) = (H^{\pm} - z)^{-1}$ are the Green's functions of $H^{\pm}$.

The inversion formula (see e.g. Reed and Simon 1972)

$$\lim_{\epsilon\to 0+} \frac{1}{2\pi i} \int_a^b [G^{\pm}(\lambda + i\epsilon) - G^{\pm}(\lambda - i\epsilon)]\, d\lambda = \tfrac{1}{2}[E^{\pm}([a, b]) + E^{\pm}((a, b))]$$

implies that $g(z_1, z_2)$ determines the measure τ and its convolution (6) uniquely. The same formula implies that the densities of these measures, if they exist, are given by

$$t(\lambda_1, \lambda_2) = \frac{1}{4\pi^2} \lim_{\epsilon_1\to 0+} \lim_{\epsilon_2\to 0+} k_2(\lambda_1 + i\epsilon_1, \lambda_2 + i\epsilon_2) \tag{9}$$

$$\alpha(\lambda) = \frac{1}{4\pi^2} \lim_{\epsilon_1\to 0+} \lim_{\epsilon_2\to 0+} k_1(\lambda + i\epsilon_1, i\epsilon_2) \tag{10}$$

where

$$k_2(z_1, z_2) = g(\bar{z}_1, z_2) + g(z_1, \bar{z}_2) - g(z_1, z_2) - g(\bar{z}_1, \bar{z}_2)$$

$$k_1(z_1, z_2) = -\int_{\mathbb{R}} k_2(z_2 - \mu, \mu - z_2)\, d\mu.$$

Now, for the operators (3) all quantities above will depend on R. In the next section we calculate $\lim_{R\to\infty} g_R(z_1, z_2)$ for non-real z_1, z_2. Since the one-to-one correspondence between the Stieltjes transforms and the respective measures is continuous with respect to convergence of the Stieltjes transform for all non-real z's and weak convergence of the respective measures, the latter result will yield weak convergence of the measures τ_R from (7). We show also that the limiting measure has a bounded density which we identify with the $R = \infty$ limit of the ILAC.

It should be noted that weak convergence of measures does not imply, generally speaking, convergence of respective densities. However we do not discuss here the legitimacy of this identification, which is widely accepted in the thoery of disordered systems and is equivalent to the legitimacy of the interchangeability of the operations $\lim_{\epsilon_{1,2}\to 0+}$ and $\lim_{R\to\infty}$ in the respective Stieltjes transform (see (9), (10)). We note only that the situation here is similar to that in statistical physics where in order to perform the infinite interaction radius limit for the magnetization (which is the derivative of the free energy) one should introduce the breaking symmetry field. In our case the imaginary part of the energy can be regarded as an analogue of the symmetry breaking field.

We remark in the conclusion of this section that we consider Gaussian W's in (3) only for the sake of brevity of our proof. In fact our final formulae (21) and (22) for the $R = \infty$ limit of the Stieltjes transform of the ILAC is valid for any independent (modulo symmetry conditions $W(x, y) = W(y, x)$) and identically distributed W's such that

$$\langle W(x, y)\rangle = 0 \qquad \langle W^2(x, y)\rangle = 1 \qquad \langle |W(x, y)|^3\rangle < \infty.$$

The respective proof can be obtained by the technique, developed by Khorunzhy and Pastur (1992).

3. The limit of infinite interaction radius for the two-point Green's functions

In this section we calculate $g_R(z_1, z_2) = \langle [G_R^+(z_1), G_R^-(z_2)](0, 0)\rangle$ in the limit $R \to \infty$. We begin with simple heuristic arguments and derive the explicit formulae (20)–(23) for $\lim_{R\to\infty} g_R(z_1, z_2)$. Afterwards we shall prove these formulae rigorously.

Our arguments are based on two relations. The first is the resolvent identity:

$$G_R^+(x, r; z_1) = G_0(x - r; z_1) - \frac{v}{R^{d/2}} \sum_{s,t\in\mathbb{Z}^d} \varphi\left(\frac{|t - s|}{R}\right) G_0(x - s; z_1) G_R^+(t, r; z_1) W(s, t) \tag{11}$$

where $G_R^+(x, r; z_1)$ and $G_0(x - r; z_1)$ are the matrix elements of $G_R^+(z_1)$ and $G_0(z_1) = (H_0 - z_1)^{-1}$, respectively. The second is the formula

$$\langle W f(W)\rangle = \langle f'(W)\rangle \tag{12}$$

which is valid for Gaussian W with zero mean and unit variance and can be easily proved by integration by parts.

Introduce the notation:

$$Q_R(x, y) = [G_R^+(z_1) G_R^-(z_2)](x, y) = \sum_{r\in\mathbb{Z}^d} G_R^+(x, r; z_1) G_R^-(r, y; z_2)$$

$$T_R(x - y) = \langle Q_R(x, y)\rangle \qquad G_R(x - y; z) = \langle G_R^\pm(x, y; z)\rangle.$$

Then $g_R(z_1, z_2) = T_R(0) = \langle Q_R(0, 0)\rangle$. Applying (11), (12) and the identity

$$\begin{aligned}\frac{\partial G_R^\pm(x, y; z)}{\partial W(s, t)} &= \mp \frac{v}{R^{d/2}} \varphi\left(\frac{|s - t|}{R}\right) \\ &\quad \times [G_R^\pm(x, s; z) G_R^\pm(t, y; z) + G_R^\pm(x, t; z) G_R^\pm(s, y; z)]\alpha(s, t)\end{aligned} \tag{13}$$

$$\alpha(s, t) = \begin{cases} 1 & s = t \\ \frac{1}{2} & s \neq t \end{cases}$$

to $T_R(x-y)$ we obtain

$$
\begin{aligned}
T_R(x-y) = {} & \sum_{r\in\mathbb{Z}^d} G_0(x-r;z_1)\langle G_R^-(r,y;z_2)\rangle \\
& + \frac{v^2}{R^d}\sum_{s,t\in\mathbb{Z}^d} G_0(x-s;z_1)\langle Q_R(s,y)G_R^+(t,t;z_1)\rangle\varphi^2\left(\frac{|t-s|}{R}\right) + \delta_1 \\
& - \frac{v^2}{R^d}\sum_{s,t\in\mathbb{Z}^d} G_0(x-s;z_1)\langle G_R^-(s,y;z_2)Q_R(t,t)\rangle\varphi^2\left(\frac{|t-s|}{R}\right) - \delta_2
\end{aligned}
\tag{14}
$$

where

$$
\delta_1 = \frac{v^2}{R^d}\sum_{s,t\in\mathbb{Z}^d} G_0(x-s;z_1)\langle Q_R(t,y)G_R^+(t,s;z_1)\rangle\varphi^2\left(\frac{|t-s|}{R}\right)
$$

$$
\delta_2 = \frac{v^2}{R^d}\sum_{s,t\in\mathbb{Z}^d} G_0(x-s;z_1)\langle Q_R(t,s)G_R^-(t,y;z_2)\rangle\varphi^2\left(\frac{|t-s|}{R}\right).
$$

Since the norm of the resolvent of any self-adjoint operator is bounded from above by $|\operatorname{Im} z|^{-1}$ we have

$$
\begin{aligned}
|\delta_1| & \leqslant \frac{C_\varphi v^2}{R^2}\sum_{s\in\mathbb{Z}^d}|G_0(s,z_1)|\left\langle\left[\sum_{t\in\mathbb{Z}^d}|Q_R(t,y)|^2\right]^{1/2}\left[\sum_{t\in\mathbb{Z}^d}|G_R^+(t,z;z_1)|^2\right]^{1/2}\right. \\
& \leqslant \frac{BC_\varphi v^2}{R^d}\langle\|Q_R\|\,\|G_R^+(z_1)\|\rangle \\
& \leqslant \frac{BC_\varphi v^2}{R^d|\operatorname{Im} z_1|^2|\operatorname{Im} z_2|}
\end{aligned}
\tag{15}
$$

where $C_\varphi = \max_{t\in(R)^d}|\varphi(t)|$ and $B = \sum_{s\in\mathbb{Z}^d}|G_0(s;z_1)|$. Our form (4) of H_0 implies exponential decay of $G_0(s;z_1)$ and therefore the finiteness of B. A similar estimate holds for δ_2. Thus $|\delta_i| \leqslant CR^{-d}$, $i=1,2$, and we can neglect δ_i in (14) when $R\to\infty$.

Now consider sums over t on the RHS of (14), i.e.

$$
\frac{1}{R^d}\sum_{t\in\mathbb{Z}^d}\varphi^2\left(\frac{|t-s|}{R}\right)G_R^+(t,t;z_1)
$$

and

$$
\frac{1}{R^d}\sum_{t\in\mathbb{Z}^d}\varphi^2\left(\frac{|t-s|}{R}\right)Q_R(t,t).
$$

It is easy to see that for IID W's in (3) and $\operatorname{Im} z \neq 0$, $G_R^+(t,t;z)$ and $Q_R(t,t)$ are bounded ergodic fields with respect to $t\in\mathbb{Z}^d$. Thus if these quantities would not depend on R we could use the ergodic theorem and replace for $R\to\infty$ these sums by the respective expectations $\langle G_R^\pm(t,t;z)\rangle \equiv G_R(0,z)$ and $\langle Q_R(t,t)\rangle \equiv T_R(0)$. Carrying out this replacement in (14) and neglecting $\delta_{1,2}$ we obtain the relations

$$
\begin{aligned}
T_R(x-y) = {} & (1-v^2T_R(0))\sum_{t\in\mathbb{Z}^d} G_0(x-t,z_1)G_R(t-y;z_2) \\
& + v^2G_R(0,z_1)\sum_{t\in\mathbb{Z}^d} G_0(x-t;z_1)T_R(t-y).
\end{aligned}
\tag{16}
$$

Treating this relation as an integral equation for the kernel $T_R(x-y)$, we can easily express this kernel through $G_0(x-y)$ and $G_R(x-y)$. In particular

$$g_R(z_1, z_2) = T_R(0) = F_R(z_1, z_2)(1 + v^2 F_R(z_1, z_2))^{-1} \tag{17}$$

where

$$F_R(z_1, z_2) = [(1 - v^2 G_R(0, z_1) G_0(z_1))^{-1} G_0(z_1) G_R(z_2)](0)$$

and $G_R(z)$ denotes the Toeplitz operator defined by the kernel $G_R(x-y; z)$.

According to Khorunzhy and Pastur (1992) (see also Khorunzhy *et al* 1992)

$$\lim_{R\to\infty} G_R(x-y; z) \equiv G(x-y; z) = G_0(x-y, z + v^2 r(z)) \tag{18}$$

uniformly with respect to x and y, where $r(z) \equiv G(0, z)$ is the unique solution of the equation

$$r(z) = \int \frac{\mathrm{d}N_0(\lambda)}{\lambda - z - v^2 r(z)} \tag{19}$$

in the class of analytic functions in $C_0 = \{z \in \mathbb{C} : |\operatorname{Im} z| \neq 0\}$ such that $\operatorname{Im} r(z) \cdot \operatorname{Im} z > 0$. $N_0(\lambda)$ in (19) denotes the DOS of the unperturbed operator H_0.

As a result we obtain that the limiting form T of the operator $T_R \equiv \langle G_R^+(z_1) G_R^-(z_2)\rangle$ is

$$T = G_0(z_1 + v^2 r(z_1)) G_0(z_2 + v^2 r(z_2))(1 - v^2 T(0))^{-1} \tag{20}$$

where

$$T(0) = \frac{F(z_1, z_2)}{1 + v^2 F(z_1, z_2)} = g(z_1, z_2) = \lim_{R\to\infty} g_R(z_1, z_2) \tag{21}$$

and

$$F(z_1, z_2) = \int \frac{\mathrm{d}N_0(\lambda)}{(\lambda - z_1 - v^2 r(z_1))(\lambda - z_2 - v^2 r(z_2))}. \tag{22}$$

It should be noted that the derived formulae (20)–(22) are rather similar to those one can obtain for the product of resolvents of the Schrödinger operators (2) in the coherent potential approximation (see e.g. Yonezawa and Morigaki 1973).

In the next section we use these formulae and (9), (10) to study the ILAC and the measure (7) as $R \to \infty$. Now we give the rigorous proof of (20)–(22). Namely, we will prove below that for any $z_1, z_2 \in C_0$

$$\lim_{R\to\infty} \sup_{x,y\in\mathbb{Z}^d} |\langle [G_R^+(z_1) G_R^-(z_2)](x, y)\rangle - T(x-y)| = 0 \tag{23}$$

where T is given by (20).

The proof is based on the moment equations method proposed by Khorunzhy and Pastur (1992). This method is similar to the well known method of correlation functions in statistical physics (see e.g. Ruelle 1969). For an infinite family of moments of the Green's functions we derive an infinite system of relations in which some terms are small for large R. Regarding this infinite system of relations as an equation in a suitable Banach space we show that it can be uniquely solved as $R \to \infty$ and the corresponding solution leads to (23).

We introduce the Banach space $\mathcal{B}$ of sequences of bounded functions of an increasing number of arguments

$$\mathcal{B} = \{f : f = (f_n^m)_{n,m=1}^{\infty}, f_n^m = f_n^m(x_n, \ldots, x_1, y_n, \ldots, y_1; x_0, y_0; x_{-1}, \ldots, x_{-m}, y_{-1}, \ldots, y_{-m})\}$$

with the norm

$$\|f\| = \sup_{n,m \geqslant 1} \sup_{X_n, Y_n} \sup_{x_0, y_0} \sup_{X^m, Y^m} |f_n^m(X_n, Y_n; x_0, y_0; X^m, Y^m)|.$$

Here and below we use the notation

$$X_n = \{x_n, \ldots, x_1\} \qquad X^m = \{x_{-1}, \ldots, x_{-m}\}$$

$$t \cup X_n = \{t, x_n, \ldots, x_1\} \qquad X^m \cup t = \{x_{-1}, \ldots, x_{-m}, t\}.$$

Let us consider the moments

$$M_n^m(X_n, Y_n; x_0, y_0; X^m, Y^m) = \left\langle \prod_{j=1}^{n} G_R^+(x_j, y_j; z_1) Q_R(x_0, y_0) \prod_{j=1}^{m} G_R^-(x_{-j}, y_{-j}; z_2) \right\rangle.$$

Using again (11)–(13) we obtain the following relations for M_n^m:

$$\begin{aligned} M_n^m(X_n, Y_n; x_0, y_0; X^m, Y^m) &= L_n^m(X_n, Y_n; x_0, y_0; X^m, Y^m) \\ &\quad + \frac{v^2}{R^d} \sum_{s,t \in \mathbb{Z}^d} \varphi^2\left(\frac{|t-s|}{R}\right) G_0(x_0 - s; z_1) \\ &\quad \times \left[M_{n+1}^m(t \cup X_n, t \cup Y_n; s, y_0; X^m, Y^m) \right. \\ &\quad \left. - M_n^{m+1}(X_n, Y_n; t, t; X^m \cup s, Y^m \cup y_0)\right] \\ &\quad + \delta_n^m(X_n, Y_n; x_0, y_0; X^m, Y^m) \end{aligned} \tag{24}$$

where

$$\begin{aligned} &L_n^m(X_n, Y_n; x_0, y_0, X^m, Y^m) \\ &\quad = \sum_{t \in \mathbb{Z}^d} G_0(x_0 - t; z_1) \left\langle \prod_{j=1}^{n} G_R^+(x_j, y_j; z_1) \prod_{j=1}^{m} G_R^-(x_{-j}, y_{-j}; z_2) G_R^-(t, y_0; z_2) \right\rangle \end{aligned}$$

and

$$|\delta_n^m(X_n, Y_n; x_0, y_0; X^m, Y^m)| \leqslant \frac{1}{R^d} \frac{2v^2 C_\varphi B}{|\operatorname{Im} z_1|^{n+1} |\operatorname{Im} z_2|^{m+1}} \left(\frac{n+1}{|\operatorname{Im} z_1|} + \frac{m+1}{|\operatorname{Im} z_2|}\right). \tag{25}$$

We define the linear operator $\mathcal{A}_R$ in $\mathcal{B}$ by

$$\begin{aligned} &(\mathcal{A}_R f)_n^m(X_n, Y_n, x_0, y_0; X^m, Y^m) \\ &\quad = \frac{v^2}{R^d} \sum_{s,t \in \mathbb{Z}^d} \varphi^2\left(\frac{|t-s|}{R}\right) G_0(x_0 - s; z_1) \\ &\quad \times \left[f_{n+1}^m(t \cup X_n, t \cup Y_n; s, y_0; X^m, Y^m) \right. \\ &\quad \left. - f_n^{m+1}(X_n, Y_n; t, t; X^m \cup s, Y^m \cup y_0)\right] \end{aligned}$$

and write (24) as the system of equations

$$M^{(R)} = \mathcal{A}_R M^{(R)} + L^{(R)} + \delta^{(R)}$$

where $M^{(R)} = (M_n^m)_{m,n=1}^{\infty}$, $\delta^{(R)} = (\delta_n^m)_{m,n=1}^{\infty}$, $L^{(R)} = (L_n^m)_{m,n=1}^{\infty}$.

Simple calculations similar to those in (15) show that $M^{(R)}$, $L^{(R)}$ and $\delta^{(R)}$ belong to $\mathcal{B}$ and $\|\mathcal{A}_R\| \leqslant 2v^2/|\operatorname{Im} z_1|$. Then, if $(z_1, z_2) \in \mathcal{D}$, where

$$\mathcal{D} = \{(z_1, z_2) \in \mathbb{C}^2 : |\operatorname{Im} z_1| > \max(4v^2, 1) \text{ and } |\operatorname{Im} z_2| > 1\}$$

the system of equations

$$m = \mathcal{A}_R m + l^{(R)} \tag{26}$$

with $l^{(R)} = (l_n^m)_{m,n=1}^{\infty}$,

$$l_n^m(X_n, Y_n; x_0, y_0; X^m, Y^m)$$
$$= \sum_{t\in\mathbb{Z}^d} G_0(x_0 - t; z_1) \prod_{j=1}^{n} \langle G_R^+(x_j, y_j; z_1)\rangle \prod_{j=1}^{m} \langle G_R^-(x_{-j}, y_{-j}; z_2)\rangle \langle G_R^-(t, y_0; z_2)\rangle$$

has a unique solution $m^{(R)}$. It follows from (25) that $\|\delta^{(R)}\| \leqslant cR^{-d}$. Besides (Khorunzhy and Pastur 1992, Khorunzhy *et al* 1992) $\lim_{R\to\infty} \|L^R - l^{(R)}\| = 0$. Since $m^{(R)} - M^{(R)} = (1 - \mathcal{A})^{-1}(l^{(R)} - L_{(R)} - \delta^{(R)})$ and $\|\mathcal{A}_R\| \leqslant 1/2$, we have

$$\lim_{R\to\infty} \|M^{(R)} - m^{(R)}\| = 0. \tag{27}$$

It is easy to check that

$$(m^{(R)})_n^m(X_n, Y_n; x_0, y_0; X^m, Y^m) = T_R(x_0 - y_0) \prod_{j=1}^{n} \langle G_R^+(x_j, y_j; z_1)\rangle \prod_{j=1}^{n} \langle G_R^-(x_{-j}, y_{-j}; z_2)\rangle \tag{28}$$

satisfies (26) if $T_R(x_0 - y_0)$ solves the equation

$$T_R(x_0 - y_0) = \frac{v^2}{R^d} \sum_{t,s\in\mathbb{Z}^d} \varphi^2\left(\frac{|t-s|}{R}\right) G_0(x_0 - s; z_1)$$
$$\times [\langle G_R^+(t, t; z_1)\rangle T_R(s - y_0) - T_R(0)\langle G_R^-(s, y_0; z_2)\rangle].$$

It follows from (5) and (19) that the limiting form of this equation for $R = \infty$ is (cf (16)).

$$T(x-y) = (1 - v^2 T(0)) \sum_{t\in\mathbb{Z}^d} G_0(x-t; z_1) G(t-y; z_2) + v^2 G(0; z_1) \sum_{t\in\mathbb{Z}^d} G_0(x-t; z_1) T(t-y)$$

and its unique solution is given by (20). Combining (27), (28) and the last observation we find that (23) is valid for all $(z_1, z_2) \in \mathcal{D}$. To finish the proof we only note that *a priori* analyticity of the Green's function up to the real axes allows us to perform the analytic continuation of the solution (20) to all $z_1, z_2 \in \mathbb{C}_0$.

4. The limit of infinite R for the ILAC

Now we calculate the infinite R limit of the two-dimensional measure $\tau_R(z)$. The density of the limiting measure τ can be found from (9). Let us first show that this density is bounded. The final expression for the infinite R limit $g(z_1, z_2)$ of the two-point Green's function $\langle [G_R^+ G_R^-](0,0)\rangle$ (21) contains the infinite R limit $r(z)$ of the one-point Green's function $\langle G_R^{\pm}(0,0,z)\rangle$. We list below some properties of $r(z)$ which we shall use. First of all $r(z)$ does not depend on the choice of the sign in (3). This is due to the symmetry of the distribution of the random variables $W(x, y)$. By the inversion formula

$$N(\lambda_1) - N(\lambda_2) = \frac{1}{\pi} \lim_{\epsilon \to 0+} \int_{\lambda_1}^{\lambda_2} \operatorname{Im} r(\lambda + i\epsilon)\, d\lambda$$

$r(z)$ uniquely determines the limit $N(\lambda)$ of the IDS $N_R(\lambda)$ of $H_R^{\pm}$ as $R \to \infty$. It is natural to call the support of $dN(\lambda)$ the limiting spectrum of the family of operators $H_R^{\pm}$ which we identify with the supports of the respective integrated densities of states. This set coincides with the closure of those points $\lambda \in R$ where $\operatorname{Im} r(\lambda + i0) > 0$ and can be described as follows.

We define two functions

$$I_1(\theta) = \int_0^{2d\mathcal{J}} \frac{dN_0(\lambda)}{\lambda - \theta} \qquad \text{and} \qquad I_2(\theta) = \int_0^{2d\mathcal{J}} \frac{dN_0(\lambda)}{(\lambda - \theta)^2} \tag{29}$$

and consider the equation

$$v^{-2} = I_2(\theta) = \int_0^{2d\mathcal{J}} \frac{dN_0(\lambda)}{(\lambda - \theta)^2}. \tag{30}$$

This equation has two solutions for any $v > v_c = [I_2(0)]^{-1/2}$. For space dimensionality $d \leqslant 4$, $v_c = 0$ and (30) has two solutions for any $v > 0$. Let us denote these solutions by θ_1 and θ_2, $\theta_1 < 0$ and $\theta_2 > 2d\mathcal{J}$. Then the limiting spectrum of both $H_R^{\pm}$ is the closed interval $[a, b]$ with $a = \theta_1 - v^2 I_1(\theta_1)$, $b = \theta_2 - v^2 I_1(\theta_2)$ if $v > v_c$, and $a = -v^2 I_1(0)$, $b = 2d\mathcal{J} - v^2 I_1(2d\mathcal{J})$ otherwise.

We shall also use the inequality

$$|r(\lambda \pm i0)| \leqslant v^{-1}. \tag{31}$$

which is valid for all λ. This equality and the identity

$$\int_0^{2d\mathcal{J}} \frac{dN_0(\lambda')}{|\lambda' - \zeta(\lambda \pm i0)|^2} = \frac{1}{v^2} \qquad \lambda \in [a, b] \tag{32}$$

where $\zeta(z) \equiv z + v^2 r(z)$, can readily be derived from (19).

Since

$$\operatorname{Im} r(\lambda \pm i0) = 0 \qquad \lambda \notin [a, b] \tag{33}$$

we have $t(\lambda_1, \lambda_2) = 0$ if λ_1 or λ_2 does not belong to the spectrum and

$$\alpha(\lambda) = \frac{1}{4\pi^2} \int_a^b t(\lambda - \mu, \mu)\, d\mu.$$

Thus we have to prove the boundedness of $t(\lambda_1, \lambda_2)$ only for $\lambda_1, \lambda_2 \in [a, b]$.

Let us introduce the self-energies

$$\zeta_j = \lambda_j + v^2 r(\lambda_j \pm \mathrm{i}0) = x_j + \mathrm{i}y_j \qquad j = 1, 2 \tag{34}$$

and denote the denominator of (21) by $\Phi(\zeta_1, \zeta_2)$,

$$\Phi(\zeta_1, \zeta_2) = 1 + v^2 \int_0^{2d\mathcal{J}} \frac{\mathrm{d}N_0(\lambda)}{(\lambda - \zeta_1)(\lambda - \zeta_2)}.$$

(9), (21) and (31) show that our problem reduces to the proof that $\Phi(\zeta_1, \zeta_2)$ is strictly positive for $\lambda_1, \lambda_2 \in [a, b]$. It follows from (32) that

$$1 = v^2 \int_0^{2d\mathcal{J}} \frac{|\lambda - \zeta_2|^2}{|\lambda - \zeta_1|^2 |\lambda - \zeta_2|^2} \,\mathrm{d}N_0(\lambda) = v^2 \int_0^{2d\mathcal{J}} \frac{|\lambda - \zeta_1|^2}{|\lambda - \zeta_1|^2 |\lambda - \zeta_2|^2} \,\mathrm{d}N_0(\lambda).$$

Therefore

$$\begin{aligned} 2\,\mathrm{Re}\,\Phi(\zeta_1, \zeta_2) = v^2 &\int_0^{2d\mathcal{J}} \frac{|\lambda - \zeta_1|^2 + |\lambda - \zeta_2|^2}{|\lambda - \zeta_1|^2 |\lambda - \zeta_2|^2} \,\mathrm{d}N_0(\lambda) \\ &+ v^2 \int_0^{2d\mathcal{J}} \frac{(\lambda - \zeta_1)(\lambda - \zeta_2) + (\lambda - \bar{\zeta}_1)(\lambda - \bar{\zeta}_2)}{|\lambda - \zeta_1|^2 |\lambda - \zeta_2|^2} \,\mathrm{d}N_0(\lambda), \end{aligned}$$

or after simple algebra

$$2\,\mathrm{Re}\,\Phi(\zeta_1, \zeta_2) = v^2 \int_0^{2d\mathcal{J}} \frac{(2\lambda - x_1 - x_2)^2 + (y_1 - y_2)^2}{[(\lambda - x_1)^2 + y_1^2][(\lambda - x_2)^2 + y_2^2]} \,\mathrm{d}N_0(\lambda). \tag{35}$$

Combining this expression with (31) we obtain the inequality

$$2\,\mathrm{Re}\,\Phi(\zeta_1, \zeta_2) \geqslant \frac{v^2}{[v + b]^4} \int_0^{2d\mathcal{J}} (2\lambda - x_1 - x_2)^2 \,\mathrm{d}N_0(\lambda).$$

Now since for any unit measure $\mathrm{d}N_0$

$$\min_{c \in R} \int (\lambda - c)^2 \,\mathrm{d}N_0(\lambda) = \int \lambda^2 \,\mathrm{d}N_0(\lambda) - \left[\int \lambda \,\mathrm{d}N_0(\lambda) \right]^2$$

and for $N_0(\lambda)$ corresponding to (4) the RHS of this relation is $2\pi d\mathcal{J}^2$ we have

$$\mathrm{Re}\,\Phi(\zeta_1, \zeta_2) \leqslant \pi d \frac{\mathcal{J}^2 v^2}{[v + b]^4}.$$

It follows from this inequality and (21) that

$$t(\lambda_1, \lambda_2) \leqslant 4|g(\lambda_1 \pm \mathrm{i}0, \lambda_2 \pm \mathrm{i}0)| \leqslant \frac{4}{\pi^3 d} \frac{[v + b]^4}{v^4 \mathcal{J}^2}$$

and we obtain the upper bound for the ILAC:

$$|\alpha(\lambda)| = \int_a^b t(\lambda - \mu, \mu) \,\mathrm{d}\mu \leqslant \frac{1}{\pi^2 d} \frac{[v + b]^4}{v^4 \mathcal{J}^2} [b - a]. \tag{36}$$

The RHS of this bound tends to infinity as $J \to 0$ or $v \to \infty$ (in the latter case this is because $b - a = O(v)$). This property of the bound reflects the genuine behaviour of the ILAC for $\mathcal{J} \to 0$ or $v \to \infty$. Indeed, basing on our expression for the Stieltjes transform, it can be shown that in both limits ($\mathcal{J} = 0$ or $v = \infty$) the ILAC equals $\delta(\lambda)$. In the next section we show that the same property is true for the ILAC corresponding to the Schrödinger operators (2).

Now we remind that $\lambda = \hbar\omega - E_g$, where E_g is the width of the forbidden band in the ideal semiconductor. (33) implies that if $\lambda < 2a(v)$, $t(\lambda - \mu) \equiv 0$ for all $\mu \in [a, b]$. Therefore, as follows from (34), $\alpha(\lambda) \equiv 0$ for all $\lambda < 2a$. This means that an interband transition is impossible if the energy of photon $\hbar\omega$ is less than $\Delta = E_g + 2a$ (a is negative). Thus the width of the energy gap for $\alpha(\lambda)$ in our model is equal to Δ. Because a vanishes as $v \to 0$, Δ is positive for sufficiently weak disorder.

One of the most important properties of the ILAC is its behaviour near the edges of the gap, i.e. for $\lambda \downarrow 2a$ in our case. According to (31) and definition of $k_2(z_1, z_2)$ if $\lambda = 2a + \epsilon$, $\epsilon > 0$, the lower and upper limits of integration in (34) are $a(v)$ and $a(v) + \epsilon$, respectively. Therefore, in the view of (36) we have

$$\alpha(\lambda) \leqslant \text{const} \cdot \epsilon \qquad \epsilon \to 0+. \tag{37}$$

This bound, however, is rather rough because as a rule the density (9) tends to zero as $\lambda_1, \lambda_2 \to a, b$. Thus, it is natural to think that $\alpha(\lambda) = o(\epsilon)$, $\epsilon \to +0$. Below we prove that

$$\alpha(\lambda) = \text{const} \cdot \epsilon^2 \qquad \epsilon \to 0+ \tag{38}$$

for $d \leqslant 4$. Relations (37) and (38) should be compared with the behaviour of the ILAC for crystalline semiconductors, i.e. for $v = 0$ in (3). In this case according to (10)

$$\alpha(\lambda) = \int_{T^d} \delta(\lambda - 2E(k))\, \mathrm{d}k \tag{39}$$

where $T^d = [0, 2\pi)^d$ is the d-dimensional torus and $E(k) = \mathcal{J} \sum_{j=1}^{d} (1 - \cos k_j)$ is the dispersion law (the symbol) of the free Laplacian H_0 in (2). Since now $a = 0$ we should look for asymptotic behaviour of (39) for $\epsilon \equiv \lambda \downarrow 0$. It is easy to find, that

$$\alpha(\lambda) = \text{const} \cdot \epsilon^{(d-2)/2}. \tag{40}$$

This asymptotic formula is well known in semiconductor physics. Comparing (40) with (41) and (42) we see that even small disorder, which we are considering in the paper, makes the ILAC smaller near the edge of the gap increasing the exponent of the polynomial asymptotic. Recall that according to Kirsch and Pastur (1990) in the case of strong disorder the ILAC is exponentially small near the edge of the gap.

To prove (38) we note first that straightforward calculations yield

$$k_2(z_1, z_2) = \frac{r(z_1) - r(\bar{z}_1)}{2\pi i} \frac{r(z_2) - r(\bar{z}_2)}{2\pi i} B(z_1, z_2) + c(z_1, z_2) \tag{41}$$

where $c(z_1, z_2) \to 0$ as $\operatorname{Im} z_1$ and $\operatorname{Im} z_2$ tend to zero and

$$B(z_1, z_2) = \frac{4z^2}{|z_1 - \bar{z}_2|^2} \frac{|1 - v^2 |F(z_1, z_2)|^2|\, |1 - v^2 F(z_1, \bar{z}_2)|^2}{|1 + v^2 F(z_1, z_2)|^2 |1 + v^2 F(z_1, \bar{z}_2)|^2}.$$

Now we use the relation $z_1 - z_2 = (1 - v^2 F(z_1, z_2)(\zeta_1 - \zeta_2)$ to obtain

$$B(z_1, z_2) = \frac{1 - v^4 |F(z_1, z_2)|^2}{|\zeta_1 - \bar{\zeta}_2|}$$

and then we write $F(z_1, z_2) = (r(z_1) - r(z_2))/(z_1 - z_2)$ and $F(z_1, \bar{z}_1) = F(z_2, \bar{z}_2) = v^2$ to obtain

$$B(z_1, z_2) = 4z^6 \frac{|F(z_1, z_2)|^2 - |F(z_1, \bar{z}_2)|^2}{(\zeta_1 - \bar{\zeta}_1)(\zeta_2 - \bar{\zeta}_2)}.$$

Since

$$\mathrm{Re}(F(z_1, z_2) - F(z_1, \bar{z}_2)) = -2 \int \frac{y_1 y_2}{\mathcal{D}} \, \mathrm{d}N_0(\lambda)$$

$$\mathrm{Im}(F(z_1, z_2) - F(z_1, \bar{z}_2)) = 2 \int \frac{(\lambda - x_1) y_2}{\mathcal{D}} \, \mathrm{d}N_0(\lambda)$$

$$\mathrm{Re}(F(z_1, z_2) + F(z_1, \bar{z}_2)) = 2 \int \frac{(\lambda - x_1)(\lambda - x_2)}{\mathcal{D}} \, \mathrm{d}N_0(\lambda)$$

$$\mathrm{Im}(F(z_1, z_2) + F(z_1, \bar{z}_2)) = 2 \int \frac{y_1 (\lambda - x_2)}{\mathcal{D}} \, \mathrm{d}N_0(\lambda)$$

where $\mathcal{D} = |\lambda - \zeta_1|^2 |\lambda - \zeta_2|^2$ and ζ_j, $j = 1, 2$, are given by (34), we arrive at

$$\frac{1}{4v^6} B(z_1, z_2) = \int \frac{\mathrm{d}N_0(\lambda)}{\mathcal{D}} \int \frac{(\lambda - x_1)^2 \, \mathrm{d}N_0(\lambda)}{\mathcal{D}} - \left[\int \frac{(\lambda - x_1) \, \mathrm{d}N_0}{\mathcal{D}} \right]^2.$$

It follows from (41) that the ILAC for our model has the form

$$\alpha(\lambda) = \int_a^b \rho(\lambda - \mu) \rho(\mu) B(\lambda - \mu, \mu) \, \mathrm{d}\mu$$

where

$$\rho(\mu) = (2\pi \mathrm{i})^{-1} [r(\lambda + \mathrm{i}0) - r(\lambda - \mathrm{i}0)] = \pi^{-1} \, \mathrm{Im} \, r(\lambda + \mathrm{i}0)$$

is the density of states of $H_R^{\pm}$ in the infinite R limit.

Since $\rho(\mu) = 0$ outside of $[a, b]$, we have

$$\alpha(2a + \epsilon) = \int_a^{a+\epsilon} \rho(2a + \epsilon - \mu) \rho(\mu) B(2a + \epsilon - \mu, \mu) \, \mathrm{d}\mu.$$

It is known (see e.g. Khorunzhy *et al* 1992) that

$$\rho(\lambda) = \mathrm{const} |a - \lambda|^{1/2} \qquad \text{as } \lambda \to a + 0.$$

Therefore if

$$B = B(a, a) = 4v^6 \left[\frac{1}{v^2} \int_0^{2d\mathcal{J}} \frac{\mathrm{d}N_0(\lambda)}{(\lambda - \theta_1)^4} - \left(\int_0^{2d\mathcal{J}} \frac{\mathrm{d}N_0(\lambda)}{(\lambda - \theta_1)^3} \right)^2 \right]$$

is finite, then $\alpha(2a + \epsilon) \sim B\epsilon^2$, as $\epsilon \to 0+$. It has been mentioned already that $\theta_1 < 0$ for $d \leqslant 4$. Hence, the asymptotic behaviour of the ILAC for these dimensions (and for three dimensions, in particular) near the gap edge has the form (38). If the strength of disorder in our model is small enough, $v < v_c = [I_2(0)]^{-1/2}$, then the same holds also in high dimensions. For $v > v_c$ the asymptotic behaviour depends on the space dimensionality.

5. The limits of the ILAC for $\mathcal{J} \to 0$ or $V \to \infty$

Let us consider the Schrödinger operators in $l^2(\mathbb{Z}^d)$:

$$H^{\pm} = H_0 \pm vq(x) \qquad x \in \mathbb{Z}^d \tag{42}$$

where H_0 is given by (4) and $q(x)$ is a family of independent identically distributed random variables. In this section we prove that for these operators the ILAC tends to $\delta(\lambda)$ as $\mathcal{J} \to 0$ or $v \to \infty$.

We start with the case of small $\mathcal{J}$ and rewrite (42) as $H^{\pm} = \mathcal{J}h_0 \pm V(x)$, $V(x) = vq(x)$. It follows from the spectral theorem that the Fourier transform of (6) is (cf (8))

$$\varphi(t) \equiv \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}t\lambda}\,\mathrm{d}A(\lambda) = \langle[\mathrm{e}^{-\mathrm{i}tH^+}\mathrm{e}^{-\mathrm{i}tH^-}](0,0)\rangle.$$

By using the Duhamel identity $\mathrm{e}^{-\mathrm{i}t(A+B)} = \mathrm{e}^{-\mathrm{i}tA} - \mathrm{i}\int_0^t \mathrm{e}^{-\mathrm{i}(t-\tau)A}B\mathrm{e}^{-\tau(A+B)}\,\mathrm{d}\tau$ we obtain that

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}tH^+}\mathrm{e}^{-\mathrm{i}tH^-} &= 1 - \mathrm{i}\mathcal{J}\int_0^t \left(\mathrm{e}^{\mathrm{i}\tau V}h_0\mathrm{e}^{-\mathrm{i}\tau H^+} + \mathrm{e}^{-\mathrm{i}\tau V}h_0\mathrm{e}^{-\mathrm{i}\tau H^-}\right)\mathrm{d}\tau \\ &\quad - \mathcal{J}^2\int_0^t\int_0^t \mathrm{e}^{-\mathrm{i}(t-\tau_1)V}h_0\mathrm{e}^{-\mathrm{i}\tau_1 H^+}\mathrm{e}^{-\mathrm{i}(t-\tau_2)V}h_0\mathrm{e}^{-\mathrm{i}\tau_2 H^-}\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2.\end{aligned}$$

Since for any self-adjoint operator A

$$|\mathrm{e}^{\mathrm{i}tA}(x,y)| \leqslant \|\mathrm{e}^{\mathrm{i}tA}\| = 1$$

we have $\varphi(t) = 1 + \delta(\mathcal{J},t)$, $|\delta(\mathcal{J},t| \leqslant 2\|h_0\|t\mathcal{J} + \|h_0\|t^2\mathcal{J}^2$. Therefore for any $t < \infty\varphi(t) \to 1$ as $\mathcal{J} \to 0$. This means that $A(\lambda)$ tends to the indicator of the interval $(0,+\infty)$ or in other words $A'(\lambda) \to \delta(\lambda)$ as $\mathcal{J} \to 0$.

Now we consider the more complicated case when $v \to \infty$, i.e. when disorder is large. We prove the statement provided there exists a bounded density of the distribution of the random variables $q(x) : P\{q(x) \in \mathrm{d}q\} = p(q)\,\mathrm{d}q$, $\max p(q) < \infty$. Namely we show that the Stieltjes transform of $A(\lambda)$:

$$K(z_1,z_2) = \int_{-\infty}^{\infty}\langle[G^+(z_1-\mu)G^-(\mu-z_2)](0,0)\rangle\,\mathrm{d}\mu$$

coincides in the limit $v \to \infty$ with

$$k(z_1,z_2) = \int_{-\infty}^{\infty}\langle[g^+(z_1-\mu)g^-(\mu-z_2)](0,0)\rangle\,\mathrm{d}\mu$$

where $g^{\pm}(z) = [\pm vq(x) - z]^{-1}$. It is easy to check that for all $vk(z_1,z_2)$ is the Stieltjes *transform of* $\delta(\lambda)$, *or more precisely*

$$\delta(\lambda) = \lim_{\epsilon_1\to 0+}\lim_{\epsilon_2\to 0+}\frac{1}{4\pi^2}[k(\lambda+\mathrm{i}\epsilon,\mathrm{i}\epsilon_2) + k(\lambda-\mathrm{i}\epsilon_1,-\mathrm{i}\epsilon_2) - k(\lambda-\mathrm{i}\epsilon_1,\mathrm{i}\epsilon_2) - k(\lambda+\mathrm{i}\epsilon_1,-\mathrm{i}\epsilon_2)]$$

and the convergence of $\alpha(\lambda)$ to $\delta(\lambda)$ will follow from the uniqueness theorem for the Stieltjes transform.

Let us prove that $|K(z_1, z_2) - k(z_1, z_2)| \to 0$ as $v \to \infty$. We include the diagonal part of H_0 into z_1 and z_2 and restrict ourselves to the case of $H_0(x, x) = 0$. It follows without the loss of generality from the resolvent identity that

$$G^{\pm}(z) = g^{\pm}(z) - g^{\pm}(z)H_0g^{\pm}(z) + g^{\pm}(z)H_0G^{\pm}(z)H_0h^{\pm}(z).$$

Therefore

$$[G^{+}(z_1)G^{-}(z_2)](0,0) = [g^{+}(z_1)g^{-}(z_2)](0,0) + \sum_{j=3}^{8} r_j(z_1, z_2)$$

where

$$r_3(z_1, z_2) = [g^{+}(z_1)H_0g^{+}(z_1)g^{-}(z_2)H_0g^{-}(z_2)](0,0)$$

$$r_4(z_1, z_2) = [g^{+}(z_1)H_0G^{+}(z_1)H_0g^{+}(z_1)g^{-}(z_1)](0,0)$$

$$r_5(z_1, z_2) = [g^{+}(z_1)g^{-}(z_2)H_0G^{-}(z_2)H_0g^{-}(z_2)](0,0)$$

$$r_6(z_1, z_2) = -[g^{+}(z_1)H_0g^{+}(z_1)g^{-}(z_2)H_0G^{-}(z_2)H_0g^{-}(z_2)](0,0)$$

$$r_7(z_1, z_2) = -[g^{+}(z_1)H_0g^{+}(z_1)g^{-}(z_2)H_0G^{-}(z_2)H_0G^{-}(z_2)H_0g^{-}(z_2)](0,0)$$

$$r_8(z_1, z_2) = [g^{+}(z_1)H_0G^{+}(z_1)H_0g^{+}(z_1)g^{-}(z_2)H_0G^{-}(z_2)H_0g^{-}(z_2)](0,0)$$

because due to the equality $H_0(x, x) = 0$,

$$r_1 = -[g^{+}(z_1)H_0g^{+}(z_1)g^{-}(z_2)](0,0)$$

and

$$r_2 = -[g^{+}(z_1)g^{-}(z_2)H_0g^{-}(z_2)]$$

are equal to zero.

To estimate $\int_{-\infty}^{\infty} r_j(z_1 - \mu, \mu - z_2)\, \mathrm{d}\mu$ we use the inequalities:

$$\left\langle \frac{1}{|zq + \mu - z|^2} \right\rangle \leqslant \frac{1}{v} \frac{\max p(q)}{\pi |\operatorname{Im} z|}. \tag{43}$$

and

$$\int_{-\infty}^{\infty} \left\langle \frac{1}{|vq + \mu - z|^2} \right\rangle \mathrm{d}\mu \leqslant \frac{1}{\pi |\operatorname{Im} z|}. \tag{44}$$

It is clear that

$$\begin{aligned} \left| \int_{-\infty}^{\infty} r_3(z_1 - \mu, \mu - z_2)\, \mathrm{d}\mu \right| &\leqslant d\mathcal{J}^2 \int_{-\infty}^{\infty} \left\langle \frac{1}{|vq + \mu - z_1||vq + \mu - z_2|} \right\rangle^2 \mathrm{d}\mu \\ &\leqslant \frac{1}{v} \frac{d\mathcal{J}^2 \max p(q)}{\pi^2 |\operatorname{Im} z_1| \, |\operatorname{Im} z_2|} \end{aligned} \tag{45}$$

and

$$\left|\int_{-\infty}^{\infty} r_6(z_1-\mu,\mu-z_2)\,\mathrm{d}\mu\right| \leqslant d\mathcal{J}\int_{-\infty}^{\infty} \|H_0G^{+}(z_1)H_0\|\left\langle\frac{1}{|vq+\mu-z_1||vq+\mu-z_2|}\right\rangle^2 \mathrm{d}\mu$$
$$\leqslant \frac{1}{v}\frac{(d\mathcal{J})^3\max p(q)}{\pi^2|\operatorname{Im} z_1||\operatorname{Im} z_2|}. \tag{46}$$

Similar estimate holds for r_7. Combining the resolvent identity and (43)–(45) we find again

$$\left|\int_{-\infty}^{\infty} r_j(z_1-\mu,\mu-z_2)\,\mathrm{d}\mu\right| \leqslant \frac{C}{v} \qquad j=4,5,8$$

where C depends only on H_0, $p(q)$ and variables z_1 and z_2. Summarizing the above arguments we conclude that $|K(z_1,z_2)-k(z_1,z_2)| \to 0$ as $v \to \infty$ and therefore $\alpha(\lambda) \to \delta(\lambda)$.

It is natural to ask whether this behaviour will remain if the distribution of the potential is not absolutely continuous. At the moment we have no analytic results concerning this question but our results (see figures 1 and 2) suggest that the answer may be negative or at least the pattern of convergence is more complicated than in the smooth case. One more support of this suggestion is given by the results of Carmona *et al* (1986) according to which the integrated density of states for the one-dimensional Anderson model with the Bernoulli distributed large potential has a singular continuous component.

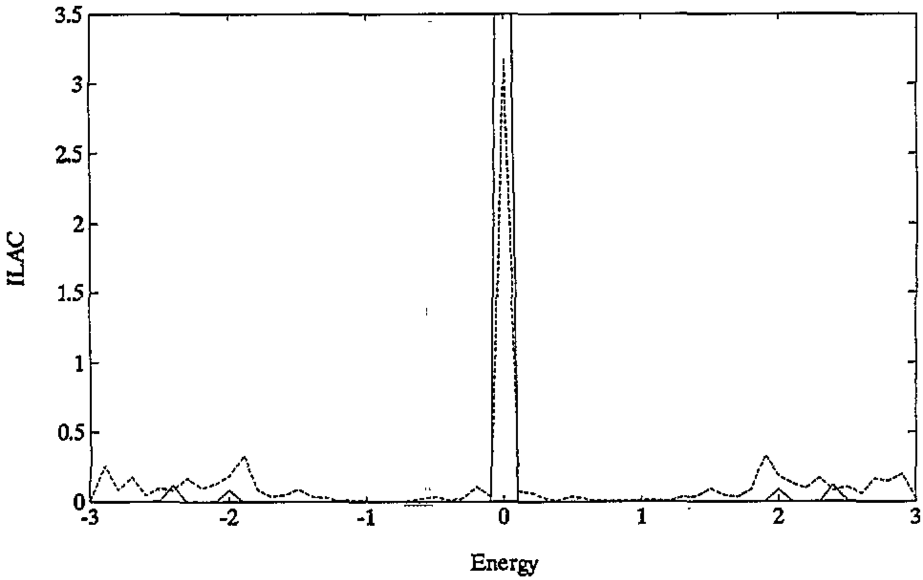

Figure 1. The result of computer simulations of the ILAC in one dimension for the case of potential with a smooth distribution. Here $H^{\pm}(42)$ are chosen to be 120×120 matrices with $\mathcal{J}=1$ and $q(x)$ are uniformly distributed over $[-1,1]$ random variables. The dashed and solid lines represent the ILAC for $v=5$ and $v=50$, correspondingly.

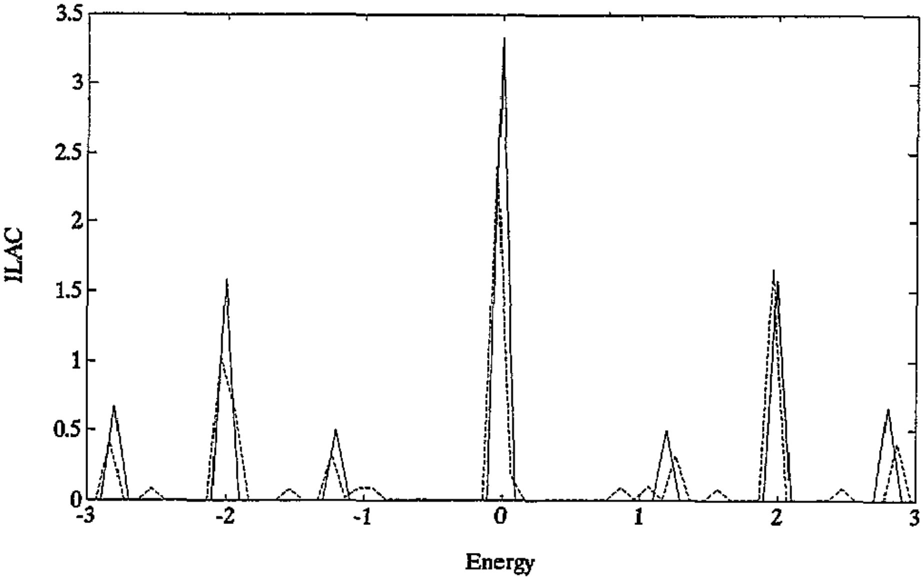

Figure 2. The result of computer simulations of the ILAC in one dimension for the case of potential with a smooth distribution. Here $H^{\pm}(42)$ are chosen to be 120×120 matrices with $\mathcal{J} = 1$ and $q(x)$ are Bernoulli random variables $(Prob\{q(x) = 1\} = Prob\{q(x) = -1\} = \frac{1}{2})$. The dashed and solid lines represent the ILAC for $v = 5$ and $v = 50$, correspondingly.

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